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A VARIATIONAL PROOF OF THE
GAUSS-BONNET FORMULA

by

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—

DEPARTMENT OF MATHEMATICS

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A variational proof of the Gauss-Bonnet formula

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0. Introduction

In [8] Peetre gave an intrinsic formula for the Euler equations for n -th order single integral variational problems. The Euler derivative introduced there is re-examined in [9] in the context of a manifold M with connection ∇ , and a new formula derived. Peetre obtained, as one of his applications, a variational approach to the Gauss-Bonnet formula for a 2-dimensional compact pseudo-Riemannian manifold equipped with the Levi-Civita connection (compare Armsen [2], where a variational proof for the 2-dimensional Riemannian case is given).

The aim of this paper is to treat, as a further application of this technique, multiple integral problems, and in particular to apply the result to Gauss-Bonnet in higher dimensions (see Allendoerfer-Weil [1], Chern [5]). The extension of the formula for the Euler derivative in [9] to general multiple integral problems is rather straightforward. However, our main concern is to find a suitable candidate for the Lagrangian in the Gauss-Bonnet situation, and the computation of the Euler derivative in that case.

Accordingly, we consider an arbitrary non-degenerate (i.e., the restriction of the metric tensor is non-degenerate) hypersurface

S in M . Let $x^1, \dots, x^{n-1}$ be local coordinates on S . It is advantageous to introduce the Grassmann algebra $\mathcal{G}$ generated by $dx^1, \dots, dx^{n-1}$. The second fundamental form on S and the restriction to S of the Riemann curvature tensor in M can then be re-interpreted as elements of the algebra $\mathcal{G} \otimes \mathcal{G}$ denoted by h and R respectively.

Next we seek the Lagrangian in the form $\varphi = F(R, h)/\sqrt{g}$, where F is a formal power series in two variables and g is the determinant of the first fundamental form (metric tensor). Let φ_{n-1} denote the term of bidegree $(n-1, n-1)$. It turns out that $\varphi_{n-1}/\sqrt{g}$ has an independent meaning and so gives an integral independent of the local coordinates.

Writing the Euler derivative in the form $A - \nabla_{X_j} B^j$, we can express A and B^j in terms of F . Imposing the requirement that $B^j = 0$, we find a quasi-unique form for the Lagrangian that involves only the Riemann curvature tensor of M . This gives the desired infinitesimal form of the Gauss-Bonnet formula. With its help, the global version is obtained by considering a C^∞ function on M with non-degenerate critical points.

The plan of our investigation is as follows. First we define the Euler derivative in the general case (§1). We consider two preliminary examples: minimal surfaces and affine minimal hypersurfaces (§2). Thereafter, the Grassmann algebra $\mathcal{G}$ is introduced, and the elements R and h are defined (§3). To streamline the computation of the Euler derivative, the next three sections (§4-6) contain the calculation of the fiber derivatives of the Lagrangian, the computation of A , and the computation of B^j , in that order. In §7 the infinitesimal form of the Gauss-Bonnet formula is derived.

The global result is obtained in the final section (§8).

The summation convention is used throughout.

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1. The Euler derivative in the general case

Let M be an n -dimensional manifold with a connection ∇ (defining torsion T and curvature R). Put $\varrho = p + p^2 + \dots + p^\nu$, where p and ν are given integers such that $1 \leq p \leq n$ and $\nu > 0$. Consider the ϱ -fold fiber product $TM \oplus \dots \oplus TM$ (ϱ times) and let φ be a C^∞ function on this fiber product.

If $X_1, \dots, X_p$ are p linearly independent commuting vector fields on M (possibly only locally defined) determining an $(n-p)$ -dimensional family of p -dimensional submanifolds of M , we can form for any of these submanifolds S the integral

$$J = \int_S \varphi(X_1, \dots, X_p, \nabla_{X_1} X_1, \nabla_{X_1} X_2, \dots, \nabla_{X_p} X_p, \nabla_{X_1}^2 X_1, \dots) dx^1 \dots dx^p,$$

where $x^1, \dots, x^p$ are local coordinates on S such that $X_1 = \frac{\partial}{\partial x^1}$, $\dots$, $X_p = \frac{\partial}{\partial x^p}$. We wish to compute the variation δJ of J .

Let Y be any vector field on M . Then we have

$$(1.1) \quad \delta J = \oint_S L_Y \varphi dx^1 \dots dx^p$$

for the variation in the direction of Y , $L_Y \varphi$ denoting the Lie derivative of φ .

Designating fiber derivatives (see [8] on this point as well as for Lie derivatives in general bundles) by $\dot{\varphi}^I$, where $I = i_1 \dots i_k$ and $1 \leq k \leq \nu$, and dropping φ 's arguments for the sake of brevity, we have the formula

$$L_Y \varphi = Y(\varphi) - \sum_I \dot{\varphi}^I (\nabla_{[Y, X_{i_1}]} \nabla_{X_{i_2}} \dots \nabla_{X_{i_{k-1}}} X_{i_k} + \dots +$$

$$+ \nabla_{X_{i_1}} \cdots \nabla_{Y, X_{i_{k-1}}} X_{i_k} + \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} (Y, X_{i_k}).$$

Using the connection we rewrite the term $Y(\varphi)$ as

$$Y(\varphi) = \nabla_Y \varphi + \sum \varphi^I (\nabla_Y \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k}),$$

thereby defining $\nabla_Y \varphi$. Observe that

$$\begin{aligned} & \nabla_Y \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} - \nabla_{Y, X_{i_1}} \nabla_{X_{i_2}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} - \cdots - \\ & \quad - \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} [Y, X_{i_k}] = \\ & = R(Y, X_{i_1}) \nabla_{X_{i_2}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} + \nabla_{X_{i_1}} \nabla_Y \nabla_{X_{i_2}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} - \\ & \quad - \nabla_{X_{i_1}} \nabla_{Y, X_{i_2}} \nabla_{X_{i_3}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} - \cdots - \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} [Y, X_{i_k}] = \\ & = \cdots = \sum_{j=1}^{k-1} \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{j-1}}} R(Y, X_{i_j}) \nabla_{X_{i_{j+1}}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} + \\ & \quad + \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} T(Y, X_{i_k}) + \nabla_{X_{i_1}} \cdots \nabla_{X_{i_k}} Y. \end{aligned}$$

Of course the R terms do not appear unless $k \geq 2$. The same comment is applicable below. The Lie derivative can thus be written as

$$\begin{aligned} L_Y \varphi &= \nabla_Y \varphi + \sum_I \varphi^I \left(\sum_{j=1}^{k-1} \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{j-1}}} R(Y, X_{i_j}) \nabla_{X_{i_{j+1}}} \cdots \nabla_{X_{i_{k-1}}} X_{i_k} + \right. \\ & \quad \left. + \nabla_{X_{i_1}} \cdots \nabla_{X_{i_{k-1}}} T(Y, X_{i_k}) + \nabla_{X_{i_1}} \cdots \nabla_{X_{i_k}} Y \right). \end{aligned}$$

(Compare to formula (1.6) in [9] .)

We now want to split $L_Y \varphi$ into two parts: one linear in Y and therefore a differential form, this is the Euler derivative, and the second consisting of terms differentiated in X_i directions. The Euler derivative, $E_Y \varphi$, is formally obtained from $L_Y \varphi$ by moving the covariant derivatives which appear to the left of Y out in front of the fiber derivatives, i.e.,

$$E_Y \varphi = \nabla_Y \varphi + \sum_I \left[\sum_{j=1}^{k-1} (-1)^{j-1} \nabla_{X_{i_{j-1}}} \dots \nabla_{X_{i_1}} \dot{\varphi}^I(R(Y, X_{i_j}) \nabla_{X_{i_{j+1}}} \dots \nabla_{X_{i_{k-1}}} X_{i_k}) + \right. \\ \left. + (-1)^{k-1} \nabla_{X_{i_{k-1}}} \dots \nabla_{X_{i_1}} \dot{\varphi}^I(T(Y, X_{i_k})) + (-1)^k \nabla_{X_{i_k}} \dots \nabla_{X_{i_1}} \dot{\varphi}^I(Y) \right].$$

This is thus a definition. (Compare [9], (1.8).) Note that for $E_Y \varphi$ we need knowledge of the behavior of $X_1, \dots, X_p$ on S only.

If Y vanishes "at infinity" we can integrate by parts in (1.1) and then end up with the formula

$$(1.2) \quad \delta J = \varepsilon \int_S E_Y \varphi dx^1 \dots dx^p.$$

At first sight, the Euler derivative appears to have serious drawbacks. We assumed that Y vanished outside a coordinate neighborhood, and we have no transformation properties for a change of local coordinates. However, if φ is a scalar density, these difficulties do not in fact arise.

Proposition 1.1. Let φ be a scalar density, i.e.,

$$\bar{\varphi}(\bar{x}(x)) \frac{d\bar{x}}{dx} = \varphi(x).$$

Here $x = (x^1, \dots, x^p)$ and $\bar{x} = (\bar{x}^1, \dots, \bar{x}^p)$ are local coordinates

on two charts that intersect, and $\frac{d\bar{x}}{dx} = \det \left(\frac{\partial \bar{x}^i}{\partial x^k} \right)$. If S is orientable and the charts are compatible, then

$$E_Y \bar{\varphi}(\bar{x}(x)) \frac{d\bar{x}}{dx} = E_Y \varphi(x)$$

for every x in the intersection and vector Y at x . Furthermore,

$$\delta J = \varepsilon \int_S E_Y \varphi dx^1 \dots dx^p$$

for arbitrary compactly supported vector fields Y defined along S .

Proof:

Let Y be a given vector field along S , and let x be a point in the intersection of our coordinate charts. Moreover, let ψ be a function with support in a compact neighborhood of x contained in the intersection.

Set $Y_1 = \psi Y$. Taking the variation of both sides of

$$\int \bar{\varphi}(\bar{x}) d\bar{x} = \int \varphi(x) dx,$$

in the direction of Y_1 , we get

$$\int L_{Y_1} \bar{\varphi}(\bar{x}) d\bar{x} = \int L_{Y_1} \varphi(x) dx.$$

Integration by parts and a change of variables gives

$$\int E_{Y_1} \bar{\varphi}(\bar{x}(x)) \frac{d\bar{x}}{dx} dx = \int E_{Y_1} \varphi(x) dx.$$

Using the fact that $Y \rightarrow E_Y \varphi$ is a form, we have

$$\int \psi (E_Y \bar{\varphi}(\bar{x}(x)) \frac{d\bar{x}}{dx} - E_Y \varphi(x)) dx.$$

Because ψ is arbitrary, we obtain

$$E_Y \bar{\varphi}(\bar{x}(x)) \frac{d\bar{x}}{dx} = E_Y \varphi(x).$$

Since Y and x also are arbitrary, the Euler derivative transforms as desired.

To prove that (1.2) is true for arbitrary compactly supported vector fields Y along S , we first note that we also could have obtained (1.2) if instead φ had had compact support in a coordinate chart. A suitable partition of unity ψ_i on S and the transformation property just proved gives

$$\delta J = \varepsilon \sum_i \int E_Y(\psi_i \varphi) dx = \varepsilon \int E_Y \varphi dx.$$

We also have the following standard

Proposition 1.2. Let φ be a scalar density. Then $E_Y \varphi(x) = 0$ if Y is tangent to S at x .

Proof:

Let $Y = \gamma^1 \frac{\partial}{\partial x^1}$ at x . Extend γ to a vector field on S with compact support in the coordinate chart. Let $\bar{x} = \bar{x}(x, \varepsilon)$ denote the one-parameter group of diffeomorphisms generated by γ , so that $\left. \frac{\partial \bar{x}}{\partial \varepsilon} \right|_{\varepsilon=0} = \gamma$. Then

$$\varphi(x, \varepsilon) = \bar{\varphi}(\bar{x}(x, \varepsilon)) \frac{d\bar{x}}{dx}(x, \varepsilon)$$

and derivation with respect to ε at $\varepsilon = 0$ yields

$$L_\gamma \varphi = \frac{\partial \varphi}{\partial x^1} \gamma^1 + \varphi \frac{\partial \gamma^1}{\partial x^1} = \frac{\partial}{\partial x^1} (\varphi \gamma^1).$$

Hence,

$$\int E_Y \varphi dx = 0,$$

and with the method of proof in the previous proposition we obtain

$$E_Y \varphi(x) = 0.$$

We now assume that M is pseudo-Riemannian, so that $T = 0$ and $\nabla g = 0$ (where g is the metric tensor). Then the T -terms in $E_Y \varphi$ drop out. If φ is a function involving g only, we also have $\nabla_Y \varphi = 0$. In the remainder of this paper we furthermore take $\nu \leq 3$.

Hence the formula for the Euler derivative for $\nu = 3$ simplifies to

$$\begin{aligned} E_Y \varphi = & -\nabla_{X_j} \dot{\varphi}^j(Y) + \dot{\varphi}^{kj}(R(Y, X_k)X_j) + \nabla_{X_j} \nabla_{X_k} \dot{\varphi}^{kj}(Y) + \\ & + \dot{\varphi}^{lkj}(R(Y, X_l) \nabla_{X_k} X_j) - \nabla_{X_l} \dot{\varphi}^{lkj}(R(Y, X_k)X_j) - \nabla_{X_j} \nabla_{X_k} \nabla_{X_l} \dot{\varphi}^{lkj}(Y). \end{aligned}$$

Putting

$$A = \dot{\varphi}^{kj}(R(\cdot, X_k)X_j) + \dot{\varphi}^{lkj}(R(\cdot, X_l) \nabla_{X_k} X_j) - \nabla_{X_l} \dot{\varphi}^{lkj}(R(\cdot, X_k)X_j)$$

$$B^j = \dot{\varphi}^j - \nabla_{X_k} \dot{\varphi}^{kj} + \nabla_{X_k} \nabla_{X_l} \dot{\varphi}^{lkj},$$

we can write this more concisely as

$$E_Y \varphi = A(Y) - \nabla_{X_j} B^j(Y).$$

For $\nu = 2$ or 1 one merely drops the $\dot{\varphi}^{lkj}$ or $\dot{\varphi}^{lkj}$ and $\dot{\varphi}^{kj}$ terms respectively.

Keeping this last comment in mind, we now discuss the case where S is orientable and has a boundary. The difference between $L_Y \Phi$ and $E_Y \Phi$ is

$$\begin{aligned} L_Y \Phi - E_Y \Phi &= X_j \left[B^j(Y) + (\dot{\Phi}^{jk} - \nabla_{X_1} \dot{\Phi}^{ljk})(\nabla_{X_k} Y) + \right. \\ &\quad \left. + \dot{\Phi}^{jkl}(\nabla_{X_k} \nabla_{X_l} Y + R(Y, X_k)X_l) \right] = \\ &= X_j (N_Y \Phi)^j, \end{aligned}$$

where the last equality defines the Noether derivative $N_Y \Phi$.

Let $g = \det g_{st} = \det g(X_s, X_t) > 0$. Assume that $\Phi/\sqrt{g}$ is a scalar, $B^j(Y)/\sqrt{g} \frac{\partial}{\partial x^j}$ is a vector, and $\dot{\Phi}^{kj}(Y)/\sqrt{g} \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^j}$ is a tensor. Then an inspection of $(N_Y \Phi)^j$ for $j \leq 2$ reveals that

$$\frac{(N_Y \Phi)^j}{\sqrt{g}} \frac{\partial}{\partial x^j}$$

is a globally defined vector field on S . Its divergence is

$$\frac{X_j (N_Y \Phi)^j}{\sqrt{g}}.$$

Thus, if $X_1, \dots, X_p$ is a positive basis on S ,

$$\delta J = \varepsilon \int_S E_Y \Phi / \sqrt{g} \, dS + \varepsilon \int_{\partial S} g(N_Y \Phi / \sqrt{g}, \mu) \, d\sigma,$$

dS and $d\sigma$ denoting the volume elements on S and ∂S respectively, and μ being the outward unit normal on ∂S .

2. Two examples

a) Minimal surfaces

Here we consider the integral (S orientable, $g > 0$)

$$J = \int_S \sqrt{g} \, dx.$$

Since $\varphi = \sqrt{g}$ does not depend on covariant derivatives, $\nu = 1$; whence $A = 0$ and $B^j = N^j = \dot{\varphi}^j$.

With g_{lm}^{co} denoting an element of the matrix of cofactors, i.e.

$\sum_m g_{jm} g_{lm}^{co} = \delta_{jl} g$, we quickly obtain

$$\begin{aligned} \dot{\varphi}^j &= \frac{1}{2} g^{-\frac{1}{2}} g_{lm}^{co} (\delta_j^l g(\cdot, X_m) + \delta_j^m g(X_l, \cdot)) = \\ &= \sqrt{g} \, g^{jm} g(\cdot, X_m). \end{aligned}$$

Use of normal coordinates gives

$$\nabla_{X_j} \dot{\varphi}^j = \sqrt{g} \, g^{jm} g(\cdot, \nabla_{X_j} X_m),$$

and ($p = \dim S$)

$$E_Y \varphi = - \nabla_{X_j} \dot{\varphi}^j(Y) = - p \, g(Y, \eta) \sqrt{g},$$

where η denotes the mean curvature normal.

We therefore have the classical formula

$$\delta J = \varepsilon \int_S - p \, g(Y, \eta) \, dS + \varepsilon \int_{\partial S} g(Y, \mu) \, d\sigma.$$

b) Affine minimal hypersurfaces (see Blaschke [4])

Let S now be a hypersurface in the $(n+1)$ -dimensional Euclidean space E . With ∇ denoting the standard flat connection

on E and $[\quad]$ determinant, we let G_{jk} be defined by

$$G_{jk} = [\nabla_{X_j} X_k, X_1, \dots, X_n].$$

Set $G = \det G_{jk}$. Assuming that $G > 0$ and that S is orientable and given an orientation, then

$$g_{jk} = \frac{G_{jk}}{G^{1/n+2}}$$

defines a pseudo-Riemannian metric on S that is invariant under volume-preserving affine transformations of E . The following obvious formulas hold:

$$G_{jk} = g_{jk} \sqrt{g} \quad \text{and} \quad \frac{G_{jk}^{co}}{G} = G^{kj} = \frac{g^{kj}}{\sqrt{g}}.$$

Let ∇^g denote the Levi-Civita connection associated with g_{jk} on S . The affine normal vector N is defined by

$$N = \frac{1}{n} g^{jk} (\nabla_{X_j} X_k - \nabla_{X_j}^g X_k).$$

(Treating S as an immersion $X: S \rightarrow E$, $N = \frac{1}{n} \Delta X$.) One easily finds that

$$(2.1) \quad [N, X_1, \dots, X_n] = \sqrt{g}.$$

Using N and the Christoffel symbols Γ_{jk}^1 of ∇^g we write $\nabla_{X_j} X_k$ as

$$(2.2) \quad \nabla_{X_j} X_k = (\Gamma_{jk}^1 + A_{jk}^1) X_1 + g_{jk} N.$$

This defines a symmetric vector-valued 2-tensor A , the difference

tensor of the connection induced on S by N , and ∇^g . One of the fundamental structural equations of this theory concerns the apolarity of A and g :

$$(2.3) \quad g^{jk} A_{jk}^1 = 0, \quad \forall 1.$$

In addition $A_{jkm} = A_{jk}^1 g_{1m}$ turns out to be symmetric in all pairs of indices. Thus even

$$(2.4) \quad A_{js}^s = 0.$$

Derivation of (2.1) and use of (2.2) and (2.4) yield

$$[\nabla_{X_j} N, X_1, \dots, X_n] = 0.$$

This means that

$$\nabla_{X_j} N = -C_{jk}^k X_k$$

is tangential. H , the mean curvature of S , is defined by

$$H = \frac{1}{n} C_s^s.$$

We are now almost ready to calculate the first variation of the volume of an affine hypersurface. Let thus $\varphi = \sqrt{g}$. The determinant $[\quad]$ is parallel and the connection ∇ has no curvature, so $\nabla \varphi = 0$ and

$$E \varphi = -\nabla_{X_j} (\dot{\varphi}^j - \nabla_{X_k} \dot{\varphi}^{kj}) = -\nabla_{X_j} B^j.$$

Here the Noether derivative is

$$(N_Y \varphi)^j = B^j(Y) + \dot{\varphi}^{jk} (\nabla_{X_k} Y).$$

The Lagrangian is $\Phi = \sqrt{g} = G^{1/n+2}$. Hence,

$$\dot{\Phi}^j = \frac{1}{n+2} G^{(1/n+2)-1} \sum_{l,m} G_{lm}^{co} \dot{G}_{lm}^j = \frac{1}{n+2} g^{lm} \dot{G}_{lm}^j.$$

Set $Y = b^1 X_1 + cN$. We find that

$$\begin{aligned} \dot{G}_{lm}^j(Y) &= [\nabla_{X_1} X_m, X_1, \dots, Y, \dots, X_n] = \\ &= g_{lm} [N, X_1, \dots, Y, \dots, X_n] - c(\Gamma_{lm}^j + A_{lm}^j) [N, X_1, \dots, X_n], \end{aligned}$$

with Y in place of X_j . Therefore, by apolarity (2.3),

$$\dot{\Phi}^j(Y) = \frac{n}{n+2} [N, X_1, \dots, Y, \dots, X_n] - \frac{1}{n+2} g^{lm} \Gamma_{lm}^j [Y, X_1, \dots, X_n].$$

In the same way,

$$\dot{\Phi}^{kj} = \frac{1}{n+2} g^{kj} [\cdot, X_1, \dots, X_n],$$

and by (2.2) and (2.4)

$$\nabla_{X_k} \dot{\Phi}^{kj} = \frac{1}{n+2} (-\Gamma_{kl}^j g^{kl} [\cdot, X_1, \dots, X_n] - [N, X_1, \dots, \underset{(j)}{\cdot}, \dots, X_n]).$$

Thus we get

$$B^j = \frac{n+1}{n+2} [N, X_1, \dots, \underset{(j)}{\cdot}, \dots, X_n],$$

and arrive at

$$\nabla_{X_j} B^j = \frac{n+1}{n+2} (C_j^j [\cdot, X_1, \dots, X_n] + \sum_{\substack{l,j \\ l \neq j}} [N, \dots, \underset{(1)}{\nabla_{X_j} X_l}, \dots, \underset{(j)}{\cdot}, \dots]).$$

The last term vanishes; therefore

$$E_Y \Phi = - \frac{n(n+1)}{n+2} H c \sqrt{g}.$$

The Noether part is

$$\frac{(N_Y \varphi)^j}{\sqrt{g}} = \frac{n+1}{n+2} b^j + \frac{1}{n+2} g^{jk} \frac{[\nabla_{X_k} Y, X_1, \dots, X_n]}{\sqrt{g}},$$

where each of the two terms are the components of vectors on S .

Now

$$\begin{aligned} \nabla_{X_k} Y &= \nabla_{X_k} (b^i X_i + cN) = \\ &= X_k(b^i)X_i + b^i \nabla_{X_k} X_i + X_k(c)N + c \nabla_{X_k} N. \end{aligned}$$

Hence,

$$\frac{[\nabla_{X_k} Y, X_1, \dots, X_n]}{\sqrt{g}} = b^i g_{ki} + X_k(c),$$

and

$$\frac{(N_Y \varphi)^j}{\sqrt{g}} = b^j + \frac{1}{n+2} g^{jk} X_k(c).$$

This yields the desired formula

$$\begin{aligned} \delta \int_S dS &= \varepsilon \int_S - \frac{n(n+1)}{n+2} Hc \, dS + \\ &+ \varepsilon \int_{\partial S} g(Y^T, \mu) \, d\sigma + \varepsilon \int_{\partial S} \frac{1}{n+2} \mu(c) \, d\sigma, \end{aligned}$$

where $Y^T = b^i X_i$.

Remark: In this remark we shall only consider orientation-preserving local coordinates on S . Then note that in this last example $\dot{\varphi}^j - \nabla_{X_k} \dot{\varphi}^{kj}$ were the components of a vector density-valued form.

This was a special instance of the following situation, which will also arise in the Gauss-Bonnet case.

Suppose that

$$\dot{\phi}^{kj} = A^{kj} \otimes \omega,$$

where A^{kj} is a tensor density on S and ω is a form along S .

By definition

$$\nabla_{X_k} \dot{\phi}^{kj} = \frac{\partial A^{kj}}{\partial x^k} \otimes \omega + A^{kj} \otimes \nabla_{X_k} \omega.$$

The expression

$$\frac{\partial A^{kj}}{\partial x^k} + \Gamma_{ks}^k A^{sj} + \Gamma_{ks}^j A^{ks} - \Gamma_{ks}^s A^{kj} = \frac{\partial A^{kj}}{\partial x^k} + \Gamma_{ks}^j A^{ks}$$

is a vector density. Hence $\nabla_{X_k} \dot{\phi}^{kj} + \Gamma_{st}^j \dot{\phi}^{st}$ is a vector density-valued form. If $\dot{\phi}^j + \Gamma_{st}^j \dot{\phi}^{st}$ is a form of the same kind, then obviously so is

$$\dot{\phi}^j - \nabla_{X_k} \dot{\phi}^{kj}.$$

3. The Grassmann algebra

Consider an orientable non-degenerate hypersurface S in an orientable n -dimensional pseudo-Riemannian manifold M . Let N be a unit vector field normal to S . We have N determine the orientation on S by specifying that a system on S is positive if it followed by N is positive on M . Let now $X_1, \dots, X_{n-1}$ be (possibly locally defined) vector fields on M such that $X_i = \frac{\partial}{\partial x^i}$ on S , where $x^1, \dots, x^{n-1}$ are orientation-preserving local coordinates on S .

Let G be the Grassmann algebra generated by $dx^1, \dots, dx^{n-1}$ (i.e., $dx^i \wedge dx^j = -dx^j \wedge dx^i$). We consider the tensor product $G \otimes G$ with multiplication given by

$$(\omega_1 \otimes \varphi_1) \wedge (\omega_2 \otimes \varphi_2) = (\omega_1 \wedge \omega_2) \otimes (\varphi_1 \wedge \varphi_2).$$

It thus has generators $\theta^i = dx^i \otimes 1$ and $\psi^j = 1 \otimes dx^j$. It is also convenient to put $\theta^{i_1 \dots i_k} = \theta^{i_1} \wedge \dots \wedge \theta^{i_k}$ and $\psi^{j_1 \dots j_l} = \psi^{j_1} \wedge \dots \wedge \psi^{j_l}$.

We confine our interest to the following two elements of $G \otimes G$:

$$R = R_{qrst} \theta^{qr} \psi^{st}$$

and

$$h = h_{st} \theta^s \psi^t,$$

where R_{qrst} are the components of the Riemann curvature tensor (restricted to S) and h_{st} are the components of the second fundamental form.

The Lagrangian we consider is

$$\Phi = F(R, h) / \sqrt{g},$$

where F is a formal power series in two variables and, moreover, $g = \det g_{st} = \det g(X_s, X_t)$. We assume that $g > 0$. Although φ takes values in a vector space, all the formal considerations of §1 are applicable. However, only the term of bidegree $(n-1, n-1)$ leads to something that has an independent meaning, so long as we keep the chosen orientation.

Let us clarify this. Denoting the term in question by φ_{n-1} , we notice that

$$\varphi_{n-1} dx^1 \dots dx^{n-1} = \varphi_{n-1} / \sqrt{g} dS,$$

dS being the surface element on S . Consider therefore

$$R^\alpha h^{n-1-2\alpha} / g$$

for $1 \leq \alpha \leq \left[\frac{n-1}{2} \right]$. It is easily verified that these expressions are invariant under a change of local coordinates. Thus $\varphi_{n-1} / \sqrt{g}$ is well-defined on all of S , independent of the local coordinates. This means that in calculating the Euler derivative we can at a given point use normal local coordinates.

4. The fiber derivatives

In this section we write down the fiber derivatives $\dot{\phi}^j$, $\dot{\phi}^{kj}$, and $\dot{\phi}^{lkj}$, with φ as in §3. For this purpose, we write

$$h = g(\nabla_{X_s} X_t, N) \theta^{st} = \frac{1}{\sqrt{g}} dV(X_1, \dots, X_{n-1}, \nabla_{X_s} X_t) \theta^{st}$$

and

$$R = 2g(\nabla_{X_q} \nabla_{X_r} X_t, X_s) \theta^{qr} \theta^{st},$$

letting dV denote the natural volume element on M .

We start with $1/\sqrt{g}$. Write $Y = b^i X_i + cN$. In example a) of §2, in the case of a hypersurface, we have shown that $(\dot{\sqrt{g}})^j(Y) = b^j \sqrt{g}$. Thus

$$(4.1) \quad \begin{cases} (1/\dot{\sqrt{g}})^j &= -(\cdot)^j/\sqrt{g} \\ (1/\dot{\sqrt{g}})^{kj} &= 0 \\ (1/\dot{\sqrt{g}})^{lkj} &= 0, \end{cases}$$

$(\cdot)^j$ designating the j :th component of the fiber derivative's variable.

Next consider h . We find using (4.1) that

$$\dot{h}^j(Y) = \frac{1}{\sqrt{g}} dV(X_1, \dots, Y, \dots, X_{n-1}, \nabla_{X_s} X_t) \theta^{st} - b^j h,$$

with Y in place of X_j . Thus

$$\dot{h}^j(Y) = -c \Gamma_{st}^j \theta^{st},$$

Γ_{st}^j denoting the Christoffel symbols for the connection induced on S by ∇ . Now the fiber derivatives of h can be listed:

$$(4.2) \quad \begin{cases} \dot{h}^j &= - \Gamma_{st}^j \theta^{st} g(\cdot, N) \\ \dot{h}^{kj} &= g(\cdot, N) \theta^{kj} \\ \dot{h}^{lkj} &= 0. \end{cases}$$

There remains R . But we find at once that

$$(4.3) \quad \begin{cases} \dot{R}^j &= R(X_q, X_r, \cdot, X_t) \theta^{qr} \delta^j_t \\ \dot{R}^{kj} &= 0 \\ \dot{R}^{lkj} &= 2g(\cdot, X_i) \theta^{lk} \delta^i_j. \end{cases}$$

Combining (4.1), (4.2), and (4.3) yields

$$(4.4) \quad \begin{cases} \dot{\phi}^j &= -(\cdot)^j \phi + \frac{1}{\sqrt{g}} \frac{\partial F}{\partial h} \dot{R}^j - \frac{1}{\sqrt{g}} \frac{\partial F}{\partial h} \Gamma_{st}^j \theta^{st} g(\cdot, N) \\ \dot{\phi}^{kj} &= \frac{1}{\sqrt{g}} \frac{\partial F}{\partial h} g(\cdot, N) \theta^{kj} \\ \dot{\phi}^{lkj} &= \frac{1}{\sqrt{g}} \frac{\partial F}{\partial h} 2g(\cdot, X_i) \theta^{lk} \delta^i_j. \end{cases}$$

5. Computation of A

Recall that

$$A(Y) = \dot{\phi}^{kj}(R(Y, X_k)X_j) - \nabla_{X_1} \dot{\phi}^{lkj}(R(Y, X_k)X_j) + \dot{\phi}^{lkj}(R(Y, X_1)\nabla_{X_k} X_j).$$

We need to compute $\nabla_{X_1} \dot{\phi}^{lkj}$. Using normal coordinates, we see that

$$X_1 g = \sum_{i,k} (X_1 g_{ik}) g_{ik}^{00} = 0.$$

Consequently,

$$\begin{aligned} \nabla_{X_1} \dot{\phi}^{lkj} &= \frac{1}{\sqrt{g}} \left[\left(\frac{\partial^2 F}{\partial R^2} (X_1 R) + \frac{\partial^2 F}{\partial R \partial h} (X_1 h) \right) 2g(\cdot, X_1) \theta^{lk, ij} + \right. \\ &\quad \left. + 2 \frac{\partial F}{\partial R} g(\cdot, \nabla_{X_1} X_i) \theta^{lk, ij} \right]. \end{aligned}$$

Note that

$$g(\cdot, \nabla_{X_1} X_i) \theta^{lk, ij} = hg(\cdot, N) \theta^{k, j}$$

in normal coordinates.

An examination of $(X_1 R) \theta^1$ and $(X_1 h) \theta^1$ is obviously in order. We have the following

Lemma 5.1. Put $R(N) = R(X_q, X_r, X_s, N) \theta^{qr, s}$. Then

$$\begin{cases} \text{i)} & (X_1 R) \theta^1 = -2hR(N) \\ \text{ii)} & (X_1 h) \theta^1 = \frac{1}{2}R(N), \end{cases}$$

in normal coordinates.

Proof:

We start with identity i). The left-hand side can be written as

$$\begin{aligned} (X_1 R) \theta^1 = & \left[(\nabla_{X_1} R)(X_q, X_r, X_s, X_t) + \right. \\ & + R(\nabla_{X_1} X_q, X_r, X_s, X_t) + R(X_q, \nabla_{X_1} X_r, X_s, X_t) + \\ & \left. + R(X_q, X_r, \nabla_{X_1} X_s, X_t) + R(X_q, X_r, X_s, \nabla_{X_1} X_t) \right] \theta^{qrl} \psi^{st}. \end{aligned}$$

A cyclical permutation of q, r, l and Bianchi's second identity reveal that the first term is zero. Even the next two terms vanish, because of the antisymmetry in θ^{qrl} versus the symmetry in $\nabla_{X_1} X_q$ and $\nabla_{X_1} X_r$. Finally, the antisymmetry in ψ^{st} and in R 's last two arguments together with $\nabla_{X_1} X_t = h_{1t} N$ yield

$$(X_1 R) \theta^1 = - 2hR(X_q, X_r, X_s, N) \theta^{qrs} \psi^s = - 2hR(N).$$

We turn to identity ii). Continued use of normal coordinates and the equation of Codazzi give

$$(X_1 h) \theta^1 = X_1 h_{st} \theta^{s1} \psi^t = \frac{1}{2} R(X_1, X_s, N, X_t) \theta^{s1} \psi^t = \frac{1}{2} R(N).$$

This lemma shows that

$$- 2(X_1 \cdot) \theta^1 = ((4h \frac{\partial}{\partial R} - \frac{\partial}{\partial h})(\cdot)) R(N)$$

when we apply this operator to a power series in R and h . Setting

$$(5.1) \quad G = 4h \frac{\partial F}{\partial R} - \frac{\partial F}{\partial h},$$

we can thus rewrite $\nabla_{X_1} \dot{\phi}^{lkj}$:

$$\nabla_{X_1} \dot{\phi}^{lkj} = \frac{1}{\sqrt{g}} \left[- \frac{\partial G}{\partial R} R(N) g(\cdot, X_1) \theta^{k,1j} + 2h \frac{\partial F}{\partial R} g(\cdot, N) \theta^{k,j} \right].$$

Now we are ready for this section's main result. Using (4.4) and the last identity above, we have

$$\begin{aligned} \dot{\phi}^{kj}(R(Y, X_k) X_j) &= \frac{1}{\sqrt{g}} \frac{\partial F}{\partial h} g(R(Y, X_k) X_j, N) \theta^{k,j}, \\ - \nabla_{X_1} \dot{\phi}^{lkj}(R(Y, X_k) X_j) &= \frac{1}{\sqrt{g}} \left[\frac{\partial G}{\partial R} R(N) g(R(Y, X_k) X_j, X_1) \theta^{k,1j} - \right. \\ &\quad \left. - 2h \frac{\partial F}{\partial R} g(R(Y, X_k) X_j, N) \theta^{k,j} \right], \end{aligned}$$

and

$$\begin{aligned} \dot{\phi}^{lkj}(R(Y, X_1) \nabla_{X_k} X_j) &= \frac{2}{\sqrt{g}} \frac{\partial F}{\partial R} g(R(Y, X_1) \nabla_{X_k} X_j, X_1) \theta^{1k,1j} = \\ &= \frac{2}{\sqrt{g}} \frac{\partial F}{\partial R} hg(R(Y, X_1) N, X_1) \theta^{1,j}. \end{aligned}$$

If we furthermore put

$$R(Y) = R(X_k, Y, X_j, X_1) \theta^{k,j1}$$

and

$$R(Y, N) = R(X_1, Y, X_1, N) \theta^{1,j1},$$

we thus find

$$A(Y) = - \frac{1}{\sqrt{g}} \left[\left(4h \frac{\partial F}{\partial R} - \frac{\partial F}{\partial h} \right) R(Y, N) + \frac{\partial G}{\partial R} R(N) R(Y) \right] = - \frac{1}{\sqrt{g}} \left[G \cdot R(Y, N) + \frac{\partial G}{\partial R} R(N) R(Y) \right].$$

6. Computation of B^j

In this section we consider F modulo terms of bidegree different from $(n-1, n-1)$. Recall that

$$B^j = \dot{\phi}^j - \nabla_{X_k} \dot{\phi}^{kj} + \nabla_{X_k} \nabla_{X_1} \dot{\phi}^{lkj}.$$

By (4.4) and the remark at the end of §2, we see that we may calculate $\dot{\phi}^j - \nabla_{X_k} \dot{\phi}^{kj}$ in normal coordinates.

We start with $\nabla_{X_k} \dot{\phi}^{kj}$.

$$\begin{aligned} \nabla_{X_k} \dot{\phi}^{kj} &= \frac{1}{\sqrt{g}} \nabla_{X_k} \left(\frac{\partial F}{\partial h} g(\cdot, N) e^{k, j} \right) = \\ &= \frac{1}{\sqrt{g}} \left[-\frac{1}{2} (4h \frac{\partial^2 F}{\partial R \partial h} - \frac{\partial^2 F}{\partial h^2}) R(N) g(\cdot, N) e^j + \frac{\partial F}{\partial h} g(\cdot, \nabla_{X_k} N) e^{k, j} \right]. \end{aligned}$$

Since

$$\frac{\partial G}{\partial h} = \frac{\partial}{\partial h} (4h \frac{\partial F}{\partial h} - \frac{\partial F}{\partial h}) = 4 \frac{\partial F}{\partial h} + 4h \frac{\partial^2 F}{\partial R \partial h} - \frac{\partial^2 F}{\partial h^2},$$

we can also write

$$\nabla_{X_k} \dot{\phi}^{kj} = \frac{1}{\sqrt{g}} \left[(2 \frac{\partial F}{\partial h} - \frac{1}{2} \frac{\partial G}{\partial h}) R(N) g(\cdot, N) e^j + \frac{\partial F}{\partial h} g(\cdot, \nabla_{X_k} N) e^{k, j} \right].$$

Turning now to $\nabla_{X_k} \nabla_{X_1} \dot{\phi}^{lkj}$, we observe that $\dot{\phi}^{lkj}$ is alternate in l and k . Thus,

$$\begin{aligned} (\nabla_{X_k} \nabla_{X_1} \dot{\phi}^{lkj})(Y) &= X_k X_1 \dot{\phi}^{lkj}(Y) - X_k \dot{\phi}^{lkj}(\nabla_{X_1} Y) - X_1 \dot{\phi}^{lkj}(\nabla_{X_k} Y) + \\ &+ \dot{\phi}^{lkj}(\nabla_{X_1} \nabla_{X_k} Y) = \frac{1}{2} \dot{\phi}^{lkj}(R(X_1, X_k) Y) = \frac{1}{\sqrt{g}} \frac{\partial F}{\partial R} R^j(Y). \end{aligned}$$

With $Y = b^i X_i + cN$ as in §4, we get

$$(6.1) \quad \dot{\phi}^j(Y) = -b^j \phi + \frac{1}{\sqrt{g}} \frac{\partial F}{\partial R} (b^i R_{qrsi} \theta^{qr} s^j + cR(N) s^j)$$

and

$$(6.2) \quad \nabla_{X_k} \dot{\phi}^{kj}(Y) = \frac{1}{\sqrt{g}} \left[c \left(2 \frac{\partial F}{\partial R} - \frac{1}{2} \frac{\partial G}{\partial h} \right) R(N) s^j - \frac{\partial F}{\partial h} b^i h_{ki} \theta^{k} s^j \right],$$

since $g(Y, \nabla_{X_k} N) = -b^i h_{ki}$. Finally,

$$(6.3) \quad \nabla_{X_k} \nabla_{X_l} \dot{\phi}^{lkj}(Y) = \frac{1}{\sqrt{g}} \frac{\partial F}{\partial R} (b^i R_{qrsi} \theta^{qr} s^j + cR(N) s^j).$$

Consequently, by (6.1), (6.2), and (6.3),

$$\begin{aligned} B^j(Y) = & -b^j \phi + \frac{1}{\sqrt{g}} b^i \left(2 \frac{\partial F}{\partial R} R_{qrsi} \theta^{qr} s^j + \frac{\partial F}{\partial h} h_{ki} \theta^{k} s^j \right) + \\ & + \frac{1}{\sqrt{g}} \left(\frac{1}{2} c \frac{\partial G}{\partial h} R(N) s^j \right). \end{aligned}$$

The following lemma will greatly simplify this expression, if we by B^j henceforth refer only to the terms of bidegree $(n-1, n-1)$.

Lemma 6.1. Let F be as above. Then

$$2 \frac{\partial F}{\partial R} R_{qrsi} \theta^{qr} s^j + \frac{\partial F}{\partial h} h_{ki} \theta^{k} s^j = \delta_{iF}^j,$$

modulo terms of bidegree different from $(n-1, n-1)$.

Proof:

It is sufficient to consider $F = R^a h^b$, $2a + b = n-1$. Then

$$2 \frac{\partial F}{\partial R} R_{qrsi} \theta^{qr} s^j + \frac{\partial F}{\partial h} h_{ki} \theta^{k} s^j = 2a R^{a-1} h^b R_{qrsi} \theta^{qr} s^j + b R^a h^{b-1} h_{ki} \theta^{k} s^j.$$

Since there are 2α ν indices in the R and β in the h , the right-hand side is essentially $R^\alpha h^\beta$, the difference being that there is an i in place of j in the lower indices. This proves the lemma.

Remark: The lemma is a special case of a general result for power series in any number of bihomogeneous commuting elements of $G \otimes G$. It generalizes the well-known determinant identity:

$$\sum_k h_{ik} h_{jk}^{co} = \delta_{ij} \det h.$$

We can now sum up this section by the formula

$$B^j = \frac{1}{\sqrt{E}} \left(\frac{\partial G}{\partial h} R(N) g(\cdot, N) \nu^j \right).$$

7. The infinitesimal form of the Gauss-Bonnet formula

Now we specialize, taking $F(R, h) = f(R)g(h)$, and we wish to determine f and h such that G will be independent of h . Because in this case, $B^j = 0$, and $\sqrt{g}A$ is a function of R only.

On the other hand, $\frac{\partial G}{\partial h} = 0$ is also a necessary condition if we want $\sqrt{g}E_Y \varphi$ to depend only on R . For working out $\nabla_{X_j} B^j$ gives familiar terms except for the derivation which falls on $R(N)$, where covariant derivatives of R come in. Now h , R , and ∇R can be chosen arbitrarily at a given point. Hence we must demand that $\frac{\partial G}{\partial h} = 0$.

From the definition of G , in (5.1), we find that

$$G = 4hf'g - fg'.$$

Thus we must have

$$4f' = \lambda f, \quad g' - \lambda hg = C,$$

where λ and C are constants. Normalizing f by $f(0) = 1$, we then get

$$\begin{aligned} f &= \exp(\lambda R/4) = \sum_0^{\infty} \frac{\lambda^k R^k}{2^k (2k)!!} \\ g &= C \exp(\lambda h^2/2) \int_a^h \exp(-\lambda t^2/2) dt = \\ &= g(0) \sum_0^{\infty} \frac{\lambda^k h^{2k}}{(2k)!!} + C \sum_0^{\infty} \frac{\lambda^k h^{2k+1}}{(2k+1)!!}. \end{aligned}$$

Observe that $g(0) = 0$ gives the formula

$$F = C \sum_{k=1}^{\infty} \lambda^{k-1} \left(\sum_{\alpha=0}^{k-1} \frac{2^{-\alpha}}{(2\alpha)!!(2k-2\alpha-1)!!} R^{\alpha} h^{2k-2\alpha-1} \right).$$

(The terms are familiar. Compare formulae (4) and (5) in §1 of Allendoerfer-Weil [1] and formula (24) in §2 of Chern [5].)

In any case, our interest is confined to the term of bidegree $(n-1, n-1)$. Therefore the parameters λ and C may be taken equal to 1. This gives

$$G = - \exp(R/4)$$

and

$$\frac{\partial G}{\partial R} = - (1/4) \exp(R/4).$$

For this choice of F , $B^j = 0$. The term in the Euler derivative of bidegree $(n-1, n-1)$, $E_Y \varphi_{n-1}$, is therefore determined by

$$A(Y) = - \frac{1}{\sqrt{8}} (G \cdot R(Y, N) + \frac{\partial G}{\partial R} R(N) R(Y)).$$

Here $R(Y, N)$ is of bidegree $(1, 1)$ in (θ, ψ) , and $R(N)R(Y)$ of bidegree $(2, 1) + (1, 2) = (3, 3)$. Hence all terms in A have odd bidegree. Consequently if n is odd, then $E_Y \varphi_{n-1} = 0$. However, if n is even, say $n = 2m$, then $n-1 = 2m-1$, and we get

$$\begin{aligned} E_Y \varphi_{n-1} &= \frac{1}{\sqrt{8}} \left[\frac{(R/4)^{m-1}}{(m-1)!} R(Y, N) + (1/4) \frac{(R/4)^{m-2}}{(m-2)!} R(N) R(Y) \right] = \\ &= \frac{1}{\sqrt{8}} \frac{1}{4^{m-1} (m-1)!} \left[R^{m-1} R(Y, N) + (m-1) R^{m-2} R(N) R(Y) \right]. \end{aligned}$$

We want to interpret the last expression in brackets. Consider

therefore the tensor product $Q' \otimes Q'$, where Q' is the Grassmann algebra generated by $dx^1, \dots, dx^{n-1}$ and an auxiliary form orthogonal to $X_1, \dots, X_{n-1}$. Let θ^j and χ^k be the corresponding generators of $Q' \otimes Q'$. Put

$$R = R(X_q, X_r, X_s, X_t) \theta^{qr} \chi^{st}$$

with $X_{2m} = N$. (We redefine $\sqrt{g}$ analogously, which does not change its value.)

The bracketed expression can now be written in a suitable form:

$$[R^{m-1}R(Y, N) + (m-1)R^{m-2}R(N)R(Y)] \theta^{2m} \chi^{2m} = \frac{1}{4m} R^m g(Y, N).$$

This can be seen as follows

$$\begin{aligned} & [R^{m-1}R(Y, N) + (m-1)R^{m-2}R(N)R(Y)] \theta^{2m} \chi^{2m} = \\ & = [R^{m-1}R(Y, N) + R^{m-2}R(N)R(Y) + R^{m-3}R(N)R \cdot R(Y) + \\ & \quad + \dots + R(N)R^{m-2}R(Y)] \theta^{2m} \chi^{2m} = \\ & = \frac{1}{2} R^{m-1} R(X_1, Y, X_j, X_i) \theta^1 \theta^{2m} \chi^{ji} = \frac{1}{4m} R^m g(Y, N). \end{aligned}$$

The first equality is just a rearrangement. The second follows because N appears in exactly one factor in each term, and

$$R(X, Y, Z, N) = \frac{1}{2}(R(X, Y, Z, N) - R(X, Y, N, Z)).$$

The final equality is just Lemma 6.1.

Consequently,

$$E_Y \varphi_{n-1} = \frac{1}{\sqrt{g}} \frac{1}{4^m m!} R^m g(Y, N).$$

Setting $\bar{\Phi} = (2\pi)^{-m} \varphi_{n-1} / \sqrt{g}$, we now have the infinitesimal Gauss-Bonnet theorem, namely

$$\delta \int_S \bar{\Phi} \, dS = \frac{1}{(2\pi)^m 2^m m!} \int_S \frac{R^m}{g} \delta N \, dS,$$

δN denoting the normal component of the variation.

8. The global Gauss-Bonnet theorem

In this final section we assume that M is compact and take g positive definite, so that M is Riemannian. We wish to prove that

$$C' \int_M R^m dV = \chi(M),$$

where $C' = ((2\pi)^m 2^{2m} m!)^{-1}$, R is expressed in an orthonormal basis at each point, and $\chi(M)$ is the Euler characteristic of M . Since we do not contribute anything essentially new, we allow ourselves to be a little bit sketchy.

We begin by taking a C^∞ function ω on M with non-degenerate critical points. (See Milnor [7] where all the background material can be found. As in Milnor's text, the torus is an especially elucidating example to have in mind.) We may assume, without loss of generality, that there is only one critical point for each critical value. Then we have

$$\chi(M) = \sum \pm 1,$$

where the summation is extended over all critical points (finite in number), and with $+$ or $-$ according to the index of $\text{grad } \omega$.

We consider the expression

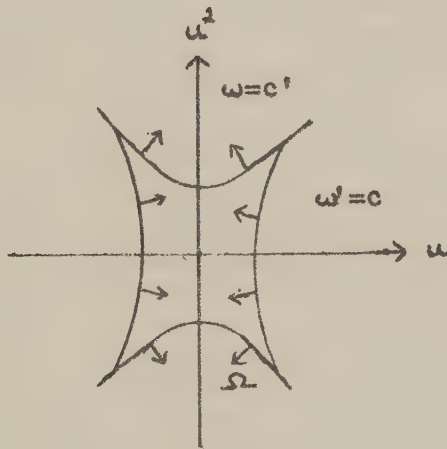
$$(8.1) \quad E = \int_{\omega=c'} \bar{\Phi} dS - \int_{\omega=c} \bar{\Phi} dS,$$

where $c' > c$ are not critical values and the hypersurfaces are oriented by $\text{grad } \omega$. If there are no critical values in the interval (c, c') , then by §7 we have

We define a continuous map Ω from $(\{\omega=c\} \cup \{\omega'=c\}) \cap U$ into the unit sphere by the formula

$$\Omega = \begin{cases} \frac{\text{grad } \omega}{|\text{grad } \omega|} & \text{on } \{\omega=c\} \cap U \\ \frac{\text{grad } \omega'}{|\text{grad } \omega'|} & \text{on } \{\omega'=c\} \cap U. \end{cases}$$

Example. If $\omega = -(u^1)^2 + (u^2)^2$, then we have the situation in the following figure:



Since S^{n-1} is oriented by the inward-pointing normal, it is clear that $-E'$ equals the degree of Ω up to a small correction. On the other hand, it is obvious that the composed map $D \circ \Omega$ has degree $+1$. Since $\det D = (-1)^\lambda$, we therefore conclude that the contribution from p is $(-1)^\lambda$, which is what we need.

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